## Algebra comutatorilor

Considerăm A și B doi operatori. Se definește

$$\boxed{\text{comutatorul}: \quad [A, B] = AB - B \cdot A}$$

$$\boxed{\text{anticomutatorul} \quad \{A, B\} = AB + BA}$$

$\hat{x}$ - operatorul de poziție: $\boxed{\hat{x} = x}$

$\hat{p}_x$ - operatorul impuls pe direcția $x$: $\boxed{\hat{p}_x = \frac{\hbar}{i}\partial_x}$

$$[\hat{x}, \hat{p}_x]\psi(x) = (\hat{x} \cdot \hat{p}_x - \hat{p}_x \cdot \hat{x})\psi(x)$$

$$= x \cdot p_x \psi(x) - p_x \cdot x\,\psi(x)$$

$$= x \cdot \frac{\hbar}{i}\partial_x \psi(x) - \frac{\hbar}{i}\partial_x (x \cdot \psi(x))$$

$$= -i\hbar\, \cancel{x \cdot \partial_x \psi(x)} + i\hbar\,\psi(x) + i\hbar\, \cancel{x \cdot \partial_x \psi(x)}$$

$$[\hat{x}, \hat{p}_x]\psi(x) = i\hbar\,\psi(x) \quad \Big| : \psi(x)$$

!!

$$\boxed{[\hat{x}, \hat{p}_x] = i\hbar} \Rightarrow \boxed{[\hat{y}, \hat{p}_y] = i\hbar \qquad [\hat{z}, \hat{p}_z] = i\hbar}$$

$$\boxed{[\hat{x}, \hat{p}_y] = 0.}$$

$$\boxed{[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}}$$

$$\delta_{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

simbolul Kronecker.

$i, j = \{x, y, z\}$.

## Proprietăți ale comutatorilor:

- antisimetrie $[A, B] = -[B, A]$
- linearitate $[A, B + C + \ldots] = [A, B] + [A, C] + \ldots$
- conjugare: $[A, B]^+ = [B^+, A^+]$.

- distributivitate $[A, BC] = [A,B]C + B[A,C]$

**Dem:** conjugare:

$$[A,B]^+ = (AB - BA)^+ = (AB)^+ - (BA)^+$$

$$= B^+A^+ - A^+B^+ = [B^+, A^+]$$

distributivitate:

$$[A, BC] = \underline{ABC} - BCA - \underset{\sim}{BAC} + BAC$$

$$(AB - BA)\cdot C \quad + B(AC - CA)$$

$$[A, BC] = [A,B]\cdot C + B[A,C]$$

$$[AB, C] = [A,C]B + A[B,C].$$

**Ex:** Să se demonstreze relația:

$$\boxed{[AB, CD] = A[B,C]D + AC[B,D] + C[A,D]B + [A,C]DB}$$

$$[AB, CD] = \underline{ABCD} - CDAB - \underset{\sim}{ACBD} + ACBD$$

$$= A[B,C]D - CDAB + \underline{ACBD} - \underline{AC}DB + ACDB$$

$$= A[B,C]D + AC[B,D] - CDAB + ACDB + CADB - CADB$$

$$[AB, CD] = A[B,C]D + AC[B,D] + C[A,D]B + [A,C]DB \quad \checkmark$$

## Relațiile de neterminare a doi comutatori

Considerăm un sist. fizic, descris de fct de undă $|\psi\rangle$.

$|\psi\rangle$ - ket

$\langle\psi|$ - bra

$$(\psi, \psi) = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x)\cdot\psi(x)\cdot dx.$$

$$\langle A\rangle = \int_{-\infty}^{\infty} \psi^*(x)\, A\, \psi(x)\, dx = \langle\psi|A|\psi\rangle$$

$$\langle B\rangle = \langle\psi|B|\psi\rangle$$

nedeterminarea operatorului $A$: $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

$$\langle A^2\rangle = \int_{-\infty}^{\infty} \psi^*(x)\, A^2\, \psi(x)\, dx = \langle\psi|A^2|\psi\rangle$$

Introducem $\delta\hat{A} = A - \langle A\rangle$ și $\delta\hat{B} = B - \langle B\rangle$.

$\delta\hat{A}\,|\psi\rangle = |\chi\rangle \qquad\qquad \delta\hat{B}\,|\psi\rangle = |\phi\rangle$

Inecuație Swartz:

$$\underline{\langle\chi|\chi\rangle\cdot\langle\phi|\phi\rangle \geqslant |\langle\chi|\phi\rangle|^2}.$$

$$\langle\chi|\chi\rangle = \langle\psi|\,\delta\hat{A}\cdot\delta\hat{A}\,|\psi\rangle = \langle\psi|(\delta\hat{A})^2|\psi\rangle = \overline{\delta A^2}$$

$$\langle\phi|\phi\rangle = \langle\psi|(\delta\hat{B})^2|\psi\rangle = \overline{\delta B^2}$$

$$\langle\chi|\phi\rangle = \langle\psi|\,\delta\hat{A}\cdot\delta\hat{B}\,|\psi\rangle = \overline{\delta A\cdot\delta B}$$

$$\overline{\delta A^2}\cdot\overline{\delta B^2} \geqslant \overline{\delta A\cdot\delta B}^{\,2}.$$

$$\delta A\cdot\delta B = (A-\bar{A})(B-\bar{B})$$

$$\overline{\delta A\cdot\delta B}^{\,2} \geqslant \frac{1}{4}\left|\langle[A,B]\rangle\right|^2$$

$$\boxed{\Delta A\cdot\Delta B \geqslant \frac{1}{2}\left|\langle[A,B]\rangle\right|}$$

Inegalitatea generalitată de tip Heisenberg.

$\Downarrow$

Principiul de nedeterminare a lui Heisenberg

$$A = \hat{x} \qquad\qquad \Delta x\cdot\Delta p_x \geqslant \frac{1}{2}\cdot\left|\langle[\hat{x},\hat{p}_x]\rangle\right|$$

$$\hat{B} = \hat{p}_x = \frac{\hbar}{i}\partial_x \qquad\qquad \geqslant \frac{1}{2}|i\hbar| = \frac{\hbar}{2}$$

$$\boxed{\Delta x\cdot\Delta p_x \geqslant \frac{\hbar}{2}}$$ – Princ. ned. Heisenberg.

Dacă doi operatori comută atunci $\Delta A\cdot\Delta B \geqslant 0$.

## Probleme

(1). Să se determine comutatorii $[x^2, p_x]$, $[x, p_x^2]$, $[x^2, p_x^2]$. folosind prop. comutatorilor. și $[x, p_x] = i\hbar$.

a) $[x^2, p_x] = [x\cdot x, p_x] = x[x,p_x] + [x,p_x]x$

$= x\cdot i\hbar + i\hbar\cdot x$

$= 2i\hbar\,\hat{x}$

b) $[x, p_x^2] = 2i\hbar \hat{p}_x$

c) $[x^2, p_x^2] = 4i\hbar \hat{x}\hat{p}_x$.

(2) Să se găsească comutatorii $[l_x, y] = ?$, $[l_x, z] = ?$, $[l_x, p_y] = ?$, $[l_x, p_z] = ?$, $[l_x, l_y]$ unde:

$l_x = y p_z - z p_y$ - reprezintă componenta $x$ a momentului cinetic.

$$[l_x, y] = [y p_z - z p_y, y]$$
$$= [y p_z, y] - [z p_y, y]$$
$$= y\underbrace{[p_z, y]}_{0} + \underbrace{[y, y]}_{0} p_z - z\underbrace{[p_y, y]}_{-i\hbar} - \underbrace{[z, y]}_{0} p_y$$

$$\boxed{[l_x, y] = i\hbar \cdot z}$$

$$\boxed{[l_x, p_y] = i\hbar p_z}$$

$$\boxed{[l_x, z] = -i\hbar y}$$

$$\boxed{[l_x, p_z] = -i\hbar p_y}$$

$$[l_x, p_y] = [y p_z - z p_y, p_y]$$
$$= [y p_z, p_y] - \underbrace{[z p_y, p_y]}_{0}$$
$$= \underbrace{[y, p_y]}_{i\hbar} p_z + y\underbrace{[p_z, p_y]}_{0}$$
$$= i\hbar p_z$$

$$\begin{cases} l_x = y p_z - z p_y \\ l_y = z p_x - x p_z \\ l_z = x p_y - y p_x \end{cases}$$

- componentele operatorului moment cinetic

$$[l_x, l_y] = [y \cdot p_z - z \cdot p_y \,,\, z \cdot p_x - x \cdot p_z]$$

$$= [y p_z, z p_x] - [z p_y, z p_x] - [y p_z, x p_z] + [z p_y, x p_z]$$

$$[y p_z, z p_x] = y [p_z, z] p_x = -i\hbar\, y p_x$$

$$-[z p_y, z p_x] = 0$$

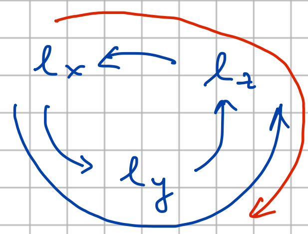
$$-[y p_z, x p_z] = 0$$

$$[z p_y, x p_z] = x p_y \cdot [z, p_z] = i\hbar\, x p_y$$

$$[l_x, l_y] = i\hbar (x \cdot p_y - y p_x) = i\hbar\, l_z.$$

$$\boxed{[l_x, l_y] = i\hbar\, l_z}$$

$$\boxed{[l_x, l_z] = -i\hbar\, l_y}$$




$$[l_x, l_y] = i\hbar\, l_z$$
$$[l_x, l_z] = -i\hbar\, l_y$$
$$[l_z, l_x] = i\hbar\, l_y$$
$$[l_y, l_z] = i\hbar\, l_x$$
$$[l_z, l_y] = -i\hbar\, l_x$$

(3) Să se studieze Hermiticitatea operatorilor $\hat{x}$, $\hat{p}_x = \frac{\hbar}{i} \frac{d}{dx}$,

$a_x = \hbar \cdot \frac{d}{dx}$

Conjugata hermitică sau adjunctul operatorului A

$A^+$ - notație ptr conj. hermitică.

$$(\phi, A\psi) = (A^+\phi, \psi)$$

$$\int_{-\infty}^{\infty} \phi^*(x)\, \hat{A} \cdot \psi(x)\, dx = \int_{-\infty}^{\infty} (A^+ \phi(x))^* \cdot \psi(x)\, dx$$

a) $A = \hat{x}$.

$$(\psi, x\psi) = \int_{-\infty}^{\infty} \psi^*(x)\left(x \cdot \psi(x)\right) dx = \int_{-\infty}^{\infty} \left(x \cdot \psi^*(x)\right) \cdot \psi(x)\, dx$$

$x \in \mathbb{R}, \quad x = x^*$

$$= \int_{-\infty}^{\infty} \left(x \cdot \psi(x)\right)^* \cdot \psi(x)\, dx = (x\psi, \psi).$$

$$(\psi, x\psi) = (x\psi, \psi) \Rightarrow \boxed{x = x^+} \Rightarrow$$ operatorul de poziție este hermitic.

b) $A = p_x = \frac{\hbar}{i} \frac{d}{dx}$

$$(\psi, p_x\psi) = \int_{-\infty}^{\infty} \psi^*(x)\; p_x \psi(x)\, dx = \int_{-\infty}^{\infty} \psi^*(x) \cdot \frac{\hbar}{i}\left(\frac{d}{dx} \psi(x)\right) dx$$

obs:

$$\int_{-\infty}^{\infty} f \cdot g'\, dx = f \cdot g \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f' \cdot g\, dx$$

$f$ și $g$ sunt funcțiile de undă $\psi^*(x), \psi(x) \Rightarrow f \cdot g \Big|_{-\infty}^{\infty} = 0$

$|\psi(x)|^2 \Big|_{-\infty}^{\infty} = 0.$

$$\underline{(\psi, p_x\psi)} = -\int_{-\infty}^{\infty} \left(\frac{\hbar}{i} \frac{d}{dx} \psi^*(x)\right) \cdot \psi(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\hbar}{i} \frac{d}{dx} \psi(x)\right)^* \cdot \psi(x)\, dx = (p_x\psi, \psi)$$

$\boxed{p_x^+ = p_x} \Rightarrow$ operatorul $p_x$ este Hermitic!

c) $A = a_x = \hbar \cdot \frac{d}{dx}$

$$\underline{(\psi, a_x\psi)} = \int_{-\infty}^{\infty} \psi^*(x) \cdot \hbar \cdot \frac{d}{dx} \psi(x)\, dx = -\int_{-\infty}^{\infty} \hbar \cdot \left(\frac{d}{dx} \psi^*(x)\right) \cdot \psi(x)\, dx$$

$$= -\int_{-\infty}^{\infty} \left(\hbar \cdot \frac{d}{dx} \psi(x)\right)^* \cdot \psi(x)\, dx = \underline{-(a_x\psi, \psi)}$$

$\Rightarrow$ $\boxed{(\Psi, a_x \Psi) = -(a_x \Psi, \Psi)}$ Un operator care satisface ec. se numește operator antihermitic. – nu are semnificație fizică.

In general conjugata hermitică nu se obține prin conjugare complexă:

$$\begin{cases} x^+ = x & \text{– hermitic} \\ \left(\frac{\hbar}{i}\frac{d}{dx}\right)^+ = \frac{\hbar}{i}\frac{d}{dx} & \text{– Hermitic} \\ \left(\frac{d}{dx}\right)^+ = -\frac{d}{dx} & \text{– antihermitic} \end{cases} \qquad \boxed{p_x^+ = p_x}$$

conjugarea complexă:

$$\begin{cases} x^* = x \\ \left(\frac{\hbar}{i}\frac{d}{dx}\right)^* = -\frac{\hbar}{i}\frac{d}{dx} \\ \left(\frac{d}{dx}\right)^* = \frac{d}{dx} \end{cases} \qquad \begin{aligned} &\Rightarrow p_x^* = -p_x \\ &a_x^* = a_x \end{aligned}$$

(4.) Să se demonstreze că $\hat{l}_x = yp_z - zp_y$ reprezintă un op. Hermitic.

$$\boxed{(AB)^+ = B^+ \cdot A^+}$$

$$l_x^+ = (yp_z - zp_y)^+ = (yp_z)^+ - (zp_y)^+$$
$$= p_z^+ y^+ - p_y^+ z^+$$

$$\begin{cases} x = x^+ \\ y = y^+ \\ z = z^+ \end{cases} \qquad \begin{cases} p_x = p_x^+ \\ p_y = p_y^+ \\ p_z = p_z^+ \end{cases} \qquad \begin{aligned} &= p_z \cdot y - p_y \cdot z \\ &= yp_z - zp_y \\ &= l_x \end{aligned}$$

$$[p_z, y] = 0$$
$$p_z y - y p_z = 0$$
$$p_z \cdot y = y \cdot p_z$$

$\Rightarrow$ $\boxed{l_x^+ = l_x}$ $\boxed{l_y^+ = l_y}$ $\boxed{l_z^+ = l_z}$

$\Rightarrow$ $\vec{l}\,(l_x, l_y, l_z)$ este un operator hermitic, reprezintă momentul cinetic.

5. Să se calculeze comutatorul $[x^m, p_x] = ?$ unde $m$ este un număr întreg $> 1$.

$$[x^m, p_x] = [\underbrace{x \cdot x \cdots \cdot x}_{m \text{ ori}}, p_x] =$$

$m=1$:

$$[x, p_x] = i\hbar$$

$m=2$:

$$[x^2, p_x] = [x \cdot x, p_x] = 2i\hbar \cdot x.$$

$m=3$:

$$[x^3, p_x] = [x \cdot x^2, p_x] = x\underbrace{[x^2, p_x]}_{2i\hbar x} + \underbrace{[x, p_x] \cdot x^2}_{i\hbar \cdot x^2}$$

$$[x^3, p_x] = 2i\hbar x^2 + i\hbar x^2 = 3i\hbar x^2.$$

$\vdots$

Presupunem că $[x^m, p_x] = m \cdot i\hbar x^{m-1}$ este adev.

$$[x^{m+1}, p_x] = (m+1) i\hbar x^m$$

$$[x \cdot x^m, p_x] = x[x^m, p_x] + [x, p_x] x^m$$
$$= x \cdot m \cdot i\hbar x^{m-1} + i\hbar x^m$$
$$= (m+1) i\hbar x^m$$

$P(m) \Rightarrow P(m+1)$

$\Rightarrow$ Folosind metoda inducției matematice $\Rightarrow$ $\boxed{[x^m, p_x] = m \cdot i\hbar x^{m-1}}$

6. Să se calculeze comutatorul : $[e^{ix}, p_x] = ?$ folosind dezvoltarea în serie Taylor a exponențialei.

$$[e^{ix}, p_x] = \left[\sum_{n=0}^{\infty} \frac{(ix)^n}{n!}, p_x\right] =$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \underbrace{[x^n, p_x]}_{i\hbar \cdot n \cdot x^{n-1}} = \sum_{n=0}^{\infty} \frac{i^n}{\cancel{n!}} \cdot i\hbar \cdot \cancel{n} \cdot x^{n-1}$$

$$= \sum_n \frac{i^{n-1}}{(n-1)!} \cdot x^{n-1} \cdot \hbar = \hbar \cdot e^{ix}$$

$$[e^{ix}, p_x] = \hbar \cdot e^{ix}$$

Se poate arată în general că:

$$\boxed{[f(x), p_x] = i\hbar \cdot f'(x)}$$

$$f(x) = x^m \Rightarrow [x^m, p_x] = i\hbar \cdot m \cdot x^{m-1}$$ ✓

Se poate demonstra că:

$$\boxed{[x, g(p_x)] = i\hbar \cdot g'(p_x)}$$

$$[x, p_x^m] = i\hbar\, m\, p_x^{m-1}$$