

Algebra comutatorilor.

Considerăm A și B doi operatori. Se definește

$$\text{comutatorul : } [A, B] = AB - B \cdot A$$

$$\text{anticomutatorul } \{A, B\} = AB + BA$$

$$\hat{x} - \text{operatorul de poziție : } \hat{x} = x$$

$$\hat{p}_x - \text{operatorul impuls pe direcția } x: \hat{p}_x = \frac{\hbar}{i} \partial_x$$

$$\begin{aligned} [\hat{x}, \hat{p}_x] \psi(x) &= (\hat{x} \cdot \hat{p}_x - \hat{p}_x \cdot \hat{x}) \psi(x) \\ &= x \cdot \hat{p}_x \psi(x) - \hat{p}_x \cdot x \psi(x) \\ &= x \cdot \frac{\hbar}{i} \partial_x \psi(x) - \frac{\hbar}{i} \partial_x (x \cdot \psi(x)) \\ &= -i\hbar x \cdot \cancel{\partial_x \psi(x)} + i\hbar \psi(x) + i\hbar x \cdot \cancel{\partial_x \psi(x)} \end{aligned}$$

$$[\hat{x}, \hat{p}_x] \psi(x) = i\hbar \psi(x) \quad | : \psi(x)$$

$$[\hat{x}, \hat{p}_x] = i\hbar$$

$$\Rightarrow [\hat{y}, \hat{p}_y] = i\hbar$$

$$[\hat{z}, \hat{p}_z] = i\hbar$$

$$[\hat{x}, \hat{p}_y] = 0.$$

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

simbolul Kronecker.

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

$$i, j = \{x, y, z\}.$$

Proprietăți ale comutatorilor :

- antisimetrie $[A, B] = -[B, A]$
- linearitate $[A, B+C+\dots] = [A, B] + [A, C] + \dots$
- conjugare: $[A, B]^+ = [B^+, A^+]$.

- distributivitate $[A, BC] = [A, B]C + B[A, C]$ -2-

Dem: conjugare:

$$[A, B]^{\dagger} = (AB - BA)^{\dagger} = (AB)^{\dagger} - (BA)^{\dagger} \\ = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} = [B^{\dagger}, A^{\dagger}]$$

distributivitate:

$$[A, BC] = \underbrace{ABC - BCA}_{(AB - BA) \cdot C} - \underbrace{BAC + CAB}_{B(A - CA)}$$

$$[A, BC] = [A, B] \cdot C + B[A, C]$$

$$[AB, C] = [A, C]B + A[B, C]$$

Ex: Să se demonstreze relația:

$$[AB, CD] = A[B, C]D + AC[B, D] + C[A, D]B + [A, C]DB$$

$$[AB, CD] = \underbrace{ABCD - CDAB}_{A[B, C]D} - \underbrace{ACBD + CADB}_{AC[B, D]} + \underbrace{ACDB + ACDB}_{C[A, D]B} + \underbrace{CADB - CADB}_{[A, C]DB}$$

$$[AB, CD] = A[B, C]D + AC[B, D] + C[A, D]B + [A, C]DB \quad \checkmark$$

Relațiile de neterminare a doi comutatori

Considerăm un sist. fizic, descris de stat de undă $|\psi\rangle$.

$|\psi\rangle$ - ket
 $\langle\psi|$ - bra

$$(\psi, \psi) = \langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(x) \cdot \psi(x) \cdot dx$$

$$\langle A \rangle = \int_{-\infty}^{\infty} \psi^*(x) A \psi(x) dx = \langle\psi|A|\psi\rangle$$

$$\langle B \rangle = \langle\psi|B|\psi\rangle$$

neterminare= operatorului A: $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$

$$\langle A^2 \rangle = \int_{-\infty}^{\infty} \psi^*(x) A^2 \psi(x) dx = \langle\psi|A^2|\psi\rangle$$

Introducem $\hat{A} = A - \langle A \rangle$ și $\hat{B} = B - \langle B \rangle$.

$$\Delta \hat{A} |\psi\rangle = |\chi\rangle$$

$$\Delta \hat{B} |\psi\rangle = |\phi\rangle$$

Inecuație Swartz:

$$\langle \chi | \chi \rangle \cdot \langle \phi | \phi \rangle \geq |\langle \chi | \phi \rangle|^2$$

$$\langle \chi | \chi \rangle = \langle \psi | \Delta \hat{A} \cdot \Delta \hat{A} | \psi \rangle = \langle \psi | (\Delta \hat{A})^2 | \psi \rangle = \overline{\Delta A^2}$$

$$\langle \phi | \phi \rangle = \langle \psi | (\Delta \hat{B})^2 | \psi \rangle = \overline{\Delta B^2}$$

$$\langle \chi | \phi \rangle = \langle \psi | \Delta \hat{A} \cdot \Delta \hat{B} | \psi \rangle = \overline{\Delta A \cdot \Delta B}$$

$$\overline{\Delta A^2} \cdot \overline{\Delta B^2} \geq \overline{\Delta A \cdot \Delta B}^2$$

$$\Delta A \cdot \Delta B = (A - \bar{A})(B - \bar{B})$$

$$\overline{\Delta A \cdot \Delta B}^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2$$

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$$

Inegalitatea generalizată de tip Heisenberg.

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Principiul de nedeterminare a lui Heisenberg

$$A = \hat{x}$$

$$\hat{B} = \hat{p}_x = \frac{\hbar}{i} \partial_x$$

$$\Delta x \cdot \Delta p_x \geq \frac{1}{2} \cdot |\langle [\hat{x}, \hat{p}_x] \rangle|$$

$$\geq \frac{1}{2} |\langle i\hbar \rangle| = \frac{\hbar}{2}$$

$$\Delta x \cdot \Delta p_x \geq \frac{\hbar}{2} \quad - \text{Princ. ned. Heisenberg.}$$

Dacă doi operatori comută atunci $\Delta A \cdot \Delta B \geq 0$.

Probleme

①. Să se determine comutatorii $[x^2, p_x]$, $[x, p_x^2]$, $[x^2, p_x^2]$, folosind prop. comutatorilor. $\Rightarrow [x, p_x] = i\hbar$.

$$a) [x^2, p_x] = \overbrace{[x \cdot x, p_x]} = x [x, p_x] + [x, p_x] x$$

$$= x \cdot i\hbar + i\hbar \cdot x$$

$$= 2i\hbar x$$

b) $[x, p_x^2] = 2i\hbar \hat{p}_x$

c) $[x^2, p_x^2] = 4i\hbar \hat{x} \hat{p}_x$

2) Să se găsească comutatorii $[l_x, y]=?$, $[l_x, z]=?$
 $[l_x, p_y]=?$ $[l_x, p_z]=?$, $[l_x, l_y]$ unde:

$l_x = y p_z - z p_y$ - reprezintă componenta x a momentului cinetic.

$$\begin{aligned}
[l_x, y] &= [y p_z - z p_y, y] \\
&= [y p_z, y] - [z p_y, y] \\
&= y [p_z, y] + [y, y] p_z - z [p_y, y] - [z, y] p_y
\end{aligned}$$

$[l_x, y] = i\hbar \cdot z$

$[l_x, p_y] = i\hbar p_z$

$[l_x, z] = -i\hbar y$

$[l_x, p_z] = -i\hbar p_y$

$$\begin{aligned}
[l_x, p_y] &= [y p_z - z p_y, p_y] \\
&= [y p_z, p_y] - [z p_y, p_y] \\
&= [y, p_y] p_z + y [p_z, p_y] - [z, p_y] p_y \\
&= i\hbar p_z + 0 - 0 \\
&= i\hbar p_z
\end{aligned}$$

$\left\{ \begin{aligned} l_x &= y p_z - z p_y \\ l_y &= z p_x - x p_z \\ l_z &= x p_y - y p_x \end{aligned} \right.$ - componentele operatorului moment cinetic

$$[l_x, l_y] = [y p_z - z p_y, z p_x - x p_z]$$

$$= [y p_z, z p_x] - [z p_y, z p_x] - [y p_z, x p_z] + [z p_y, x p_z]$$

$$\underbrace{[y p_z, z p_x]}_{\substack{\uparrow \\ \uparrow}} = y [p_z, z] p_x = -i\hbar y p_x$$

$$- [z p_y, z p_x] = 0$$

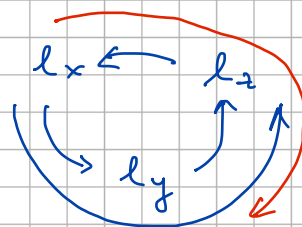
$$- [y p_z, x p_z] = 0$$

$$\underbrace{[z p_y, x p_z]}_{\substack{\uparrow \\ \uparrow}} = x p_y \cdot [z, p_z] = i\hbar x p_y$$

$$[l_x, l_y] = i\hbar (x p_y - y p_x) = i\hbar l_z$$

$$[l_x, l_y] = i\hbar l_z$$

$$[l_x, l_z] = -i\hbar l_y$$



$$[l_x, l_y] = i\hbar l_z$$

$$[l_x, l_z] = -i\hbar l_y$$

$$[l_z, l_x] = i\hbar l_y$$

$$[l_y, l_z] = i\hbar l_x$$

$$[l_z, l_y] = -i\hbar l_x$$

② Să se studieze Hermiticitatea operatorilor \hat{x} , $p_x = \frac{\hbar}{i} \frac{d}{dx}$,
 $a_x = \hbar \cdot \frac{d}{dx}$

Conjugata hermitică sau adjunctul operatorului A

A^+ - notație ptr conj. hermitică.

$$(\phi, A\psi) = (A^+\phi, \psi)$$

$$\int_{-\infty}^{\infty} \phi^*(x) \hat{A} \cdot \psi(x) dx = \int_{-\infty}^{\infty} (A^+ \phi(x))^* \cdot \psi(x) dx$$

$$a) \quad A = \hat{x}$$

$$\begin{aligned}
 (\psi, x\psi) &= \int_{-\infty}^{\infty} \psi^*(x) (x \cdot \psi(x)) dx = \int_{-\infty}^{\infty} (x \cdot \psi^*(x)) \cdot \psi(x) dx \\
 x \in \mathbb{R}, \quad x = x^* & \\
 &= \int_{-\infty}^{\infty} (x \cdot \psi(x))^* \cdot \psi(x) dx = (x\psi, \psi)
 \end{aligned}$$

$$(\psi, x\psi) = (x\psi, \psi) \Rightarrow \boxed{x = x^+}$$

$$b) \quad A = p_x = \frac{\hbar}{i} \frac{d}{dx}$$

$$(\psi, p_x \psi) = \int_{-\infty}^{\infty} \psi^*(x) p_x \psi(x) dx = \int_{-\infty}^{\infty} \psi^*(x) \cdot \frac{\hbar}{i} \left(\frac{d}{dx} \psi(x) \right) dx$$

$$\int_{-\infty}^{\infty} f \cdot g' dx = f \cdot g \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f' \cdot g dx$$

f și g sunt funcțiile de undă $\psi^*(x), \psi(x) \Rightarrow f \cdot g \Big|_{-\infty}^{\infty} = 0$
 $|\psi(x)|^2 \Big|_{-\infty}^{\infty} = 0$

$$\begin{aligned}
 \underline{(\psi, p_x \psi)} &= - \int_{-\infty}^{\infty} \left(\frac{\hbar}{i} \frac{d}{dx} \psi^*(x) \right) \cdot \psi(x) dx \\
 &= \int_{-\infty}^{\infty} \left(\frac{\hbar}{i} \frac{d}{dx} \psi(x) \right)^* \cdot \psi(x) dx = \underline{(p_x \psi, \psi)}
 \end{aligned}$$

$$\boxed{p_x^+ = p_x} \Rightarrow \text{operatorul } p_x \text{ este Hermitic}$$