

## Curs 10

### Momentul cinetic de spin

- momentul cinetic de spin (spinul) reprezintă un moment cinetic propriu a unei particule **fară nicio asociere cu fizica clasică**. Spinul reprezintă o proprietate pur cuantică a particulelor elementare.

$e^-$  - are un spin  $S = \frac{1}{2}$ .

$p, n \rightarrow S = \frac{1}{2}$  ; fotoni  $S = 1$ ;

există particule cu spin zero.

### Experimentul Stern-Gerlach (1922)

- a confirmat experimental existența spinului electronic folosind atomi de Ag.

Structura  $e^-$ :  $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10} 4p^6 4d^{10} | \underline{\underline{5s^1}}$

orbitalii de tip s au o structură sferică.

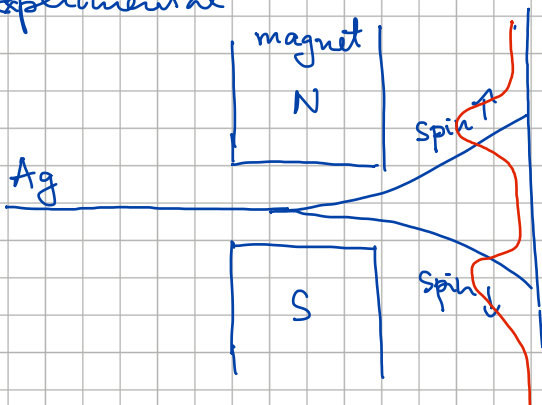
orbitalii s corespund la un număr cuantic orbital

$$l = 0.$$

Ptr un nr cuantic  $l$  ; componenta  $z$  :  $m$  ia  $2l+1$  valori.

ptr  $l=0 \Rightarrow$  experimental s-ar observa o singură valoare

Experimental



$$2s + 1 = 2$$

$$s = \frac{1}{2}$$

Spinul  $e^-$  a fost postulat ca fiind o proprietate a electronului ptr a putea explica exp. SG.

Spinul ca operator nu poate fi descris de un operator diferentia

In general spinul unei particule elementare poate lua valori întregi sau semîntregi.

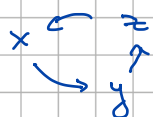
Particule cu spin semi-întreg  $S = \frac{1}{2}, \frac{3}{2}, \dots$  se numesc fermioni (electron, proton, neutron, etc)

Particulele cu spin întreg  $S = 0, 1, 2, \dots$  se numesc bosoni ( $\pi$ , foton etc)

### Teoria generală a spinului electronic ( $S = \frac{1}{2}$ )

In analogie cu teoria generală a momentului cinetic spinul  $\vec{S}$  reprezintă un operator cu trei componente  $S_x, S_y, S_z$  care satisfac relațiile de comutare

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y$$



Vectorii și valorile proprii satisfac rel. generale

$$S^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$

$$S_z |s, m_s\rangle = \hbar m_s |s, m_s\rangle$$

$$\langle s, m_s | s', m_{s'} \rangle = \delta_{ss'} \delta_{m_s m_{s'}}$$

Ptr particule cu spin  $S = \frac{1}{2}$ , numărul cuantic  $m_s$  poate lua 2 valori  $m_s = \left\{ \frac{1}{2}, -\frac{1}{2} \right\}$ .

$$-S < m_s < S$$

$$|s, m_s\rangle = \left\{ \begin{array}{l} \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \end{array} \right\}$$

$\uparrow \qquad \qquad \uparrow$   
 $s \qquad \qquad m_s$

$$S^2 \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \hbar^2 \cdot \frac{1}{2} \left( \frac{1}{2} + 1 \right) \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$S^2 \cdot \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{3}{4} \hbar^2 \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$S^2 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \frac{3}{4} \hbar^2 \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

$$S_z \left| \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{1}{2} \hbar \left| \frac{1}{2}, \frac{1}{2} \right\rangle$$

$$\uparrow$$

$$m_s = \frac{1}{2}$$

$$S_z \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = -\frac{1}{2} \hbar \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

Reprezentarea matricială :

$S^2$	$\left  \frac{1}{2}, \frac{1}{2} \right\rangle$	$\left  \frac{1}{2}, -\frac{1}{2} \right\rangle$
$\left\langle \frac{1}{2}, \frac{1}{2} \right $	$\frac{3}{4} \hbar^2$	0
$\left\langle \frac{1}{2}, -\frac{1}{2} \right $	0	$\frac{3}{4} \hbar^2$

$$S^2 = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S^2 = \frac{3}{4} \hbar^2 \cdot \mathbb{1}_2$$

$S_+$	$\left  \frac{1}{2}, \frac{1}{2} \right\rangle$	$\left  \frac{1}{2}, -\frac{1}{2} \right\rangle$
$\left\langle \frac{1}{2}, \frac{1}{2} \right $	0	$\hbar$
$\left\langle \frac{1}{2}, -\frac{1}{2} \right $	0	0

$$S_+ = \hbar \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_+ |s m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)} |s, m_s+1\rangle$$

$$S_- |s m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s-1)} |s, m_s-1\rangle$$

$S_-$	$\left  \frac{1}{2}, \frac{1}{2} \right\rangle$	$\left  \frac{1}{2}, -\frac{1}{2} \right\rangle$
$\left\langle \frac{1}{2}, \frac{1}{2} \right $	0	0
$\left\langle \frac{1}{2}, -\frac{1}{2} \right $	$\hbar$	0

$$S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \frac{1}{2} (S_+ + S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i} (S_+ - S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$S_z$	$ \frac{1}{2}, \frac{1}{2}\rangle$	$ \frac{1}{2}, -\frac{1}{2}\rangle$
$\langle \frac{1}{2}, \frac{1}{2}  $	$\frac{\hbar}{2}$	0
$\langle \frac{1}{2}, -\frac{1}{2}  $	0	$-\frac{\hbar}{2}$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In reprezentarea matricială asociem vectorii proprii

$$|\frac{1}{2}, \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\frac{1}{2}, -\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Completitudinea:  $\sum_{m_s = -\frac{1}{2}}^{\frac{1}{2}} |s, m_s\rangle \langle s, m_s| = \mathbb{I}_2$

$$|\frac{1}{2}, \frac{1}{2}\rangle \langle \frac{1}{2}, \frac{1}{2}| + |\frac{1}{2}, -\frac{1}{2}\rangle \langle \frac{1}{2}, -\frac{1}{2}| = \mathbb{I}_2$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (1 \ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot (0 \ 1) = \mathbb{I}_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_2 \quad \checkmark \quad (\text{relația de completitudine este satisfăcută})$$

Ortonormarea:

$$\langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle = (1 \ 0) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad \checkmark$$

$$\langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, -\frac{1}{2} \rangle = (1 \ 0) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad \checkmark$$

$$\langle \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle = (0 \ 1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad \checkmark$$

$$\langle \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, -\frac{1}{2} \rangle = (0 \ 1) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \quad \checkmark$$

$$\langle s m_s | s' m_s' \rangle = \delta_{ss'} \delta_{m_s m_s'}$$

Este convenabil să introducem matricile Pauli definite

$$\vec{S} = \frac{\hbar}{2} \cdot \vec{\sigma}$$

$\vec{S}$  - spinul particulei  
 $\vec{\sigma}$  - matricile Pauli.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

↳ Matricile Pauli.

Matricile Pauli au o multitudine de proprietati:

$$\sigma_j^2 = \mathbb{1}_2, \quad j = x, y, z$$

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 0; \quad j \neq k \quad (\text{anticomutator})$$

$$\{\sigma_j, \sigma_k\} = 2\mathbb{1} \cdot \delta_{jk} \quad - \text{anticomutare}$$

$$[\sigma_j, \sigma_k] = 2i \varepsilon_{jke} \sigma_e$$

$$\varepsilon_{jke} = \begin{cases} 1, & \sigma(jke) = 1 \\ -1, & \sigma(jke) = -1 \\ 0 & j \neq k, \text{ sau } j=l \text{ sau } k=l \end{cases} \quad (\text{tensorul Levi-Civita})$$

$$\sigma_j^\dagger = \sigma_j \quad j = x, y, z$$

$$\text{Tr}(\sigma_j) = 0 \quad j = x, y, z$$

$$\det(\sigma_j) = -1, \quad j = x, y, z$$

$$\sigma_x \cdot \sigma_y \cdot \sigma_z = i \mathbb{1}_2$$

Probleme:

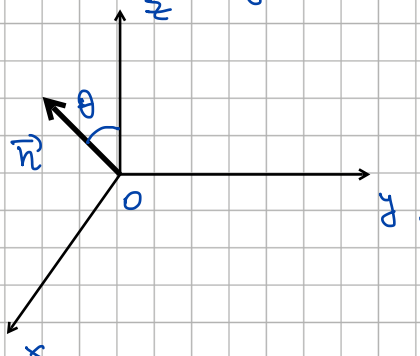
1. Să se găsească vectorii și valorile proprii ale operatorului de spin  $\vec{S}$  când acesta este orientat după o direcție  $\vec{n}$   
 $\vec{n} = \sin\theta \vec{i} + \cos\theta \vec{k}$  în planul  $Oxz$ .

$$\vec{n} \cdot \vec{S}$$

$$\vec{n} \cdot \vec{S} = (\sin\theta \vec{i} + \cos\theta \vec{k}) \cdot \frac{\hbar}{2} (\sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k})$$

$$\vec{n} = \sin\theta \vec{i} + \cos\theta \vec{k}$$

$$\vec{S} = \frac{\hbar}{2} (\sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k})$$



$$\vec{n} \cdot \vec{S} = \frac{\hbar}{2} (\sigma_x \sin\theta + \sigma_z \cos\theta)$$

$$\vec{n} \cdot \vec{S} = \frac{\hbar}{2} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sin\theta + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cos\theta \right\}$$

$$\vec{n} \cdot \vec{S} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Am reprezentat matricial operatorul  $\vec{n} \cdot \vec{S}$ .

La matricea de reprezentare dorim să determinăm valorile proprii și vectorii proprii.

$$\frac{\hbar}{2} \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \cos^2\theta - \sin^2\theta = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm \frac{\hbar}{2}$$

valorile proprii  $\lambda = \left\{ \frac{\hbar}{2}, -\frac{\hbar}{2} \right\}$ .

