Curs 10

Momentul cinetic de spin

- momentul cinetic de spin (spinul) reprezintă un moment cinetic propriu a unei particule fără nicio asociere cu fizica clasică. Spinul reprezintă o proprietate pur cuantică a particulelor elementare.

e^- - are un spin $S = \frac{1}{2}$.

p, n $\rightarrow$ $S = 1/2$; fotoni $S = 1$;

există particule cu spin zero.

Experimentul Stern-Gerlach (1922)

- a confirmat experimental existența spinului electronic folosind atomi de Ag.

$_{47}Ag$

Structura e^- : $\underbrace{1s^2\ 2s^2\ 2p^6}_{10}\ \underbrace{3s^2\ 3p^6\ 4s^2}_{10}\ \underbrace{3d^{10}}_{30}\ \underbrace{4p^6\ 4d^{10}}_{46}\ \Big|\ \underline{\underline{5s^1}}$

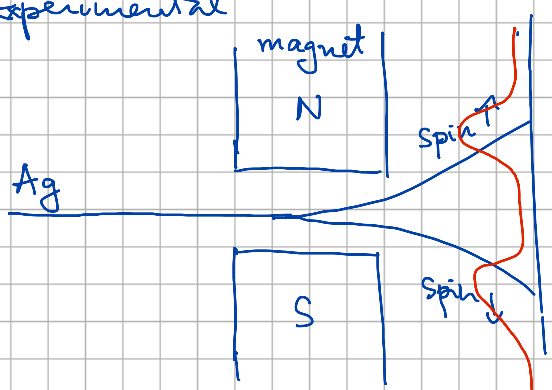
orbitalii de tip s au o structură sferică.

orbitalii s corespund la un număr cuantic orbital $l = 0$.

Ptr un nr cuantic l ; componenta z : m ia $2l+1$ valori.

Ptr $l = 0$ $\Rightarrow$ experimental s-ar observa o singură valoare

Experimental



Spinul e^- a fost postulat ca fiind o proprietate a electronului ptr a putea explica exp. SG.

Spinul ca și operator nu poate fi descris de un operator diferențial.

$2s+1 = 2$

$s = \frac{1}{2}$

În general spinul unei particule elementare poate lua valori <u>întregi</u> sau <u>semiîntregi</u>.

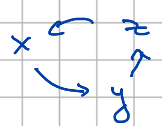
Particule cu spin semi-întreg $s = \frac{1}{2}, \frac{3}{2}, \dots$ se numesc fermioni (electron, proton, neutron, etc)

Particulele cu spin întreg $s = 0, 1, 2 \dots$ se numesc bosoni (π, foton etc)

<u>Teoria generală a spinului electronic</u> ($S = \frac{1}{2}$)

În analogie cu teoria generală a momentului cinetic spinul $\vec{S}$ reprezintă un operator cu trei componente S_x, S_y, S_z care satisfac relațiile de comutare

$$[S_x, S_y] = i\hbar S_z \quad , [S_y, S_z] = i\hbar S_x \quad , \quad [S_z, S_x] = i\hbar S_y$$



Vectorii și valorile proprii satisfac rel. generale

$$S^2 |s, m_s\rangle = \hbar^2 s(s+1) |s, m_s\rangle$$

$$S_z |s, m_s\rangle = \hbar \cdot m_s |s, m_s\rangle$$

$$\langle s, m_s | s' m_{s'}\rangle = \delta_{ss'} \delta_{m_s m_{s'}}$$

Ptr particule cu spin $S = \frac{1}{2}$, numărul cuantic m_s poate lua 2 valori $m_s = \{-\frac{1}{2}, \frac{1}{2}\}$.

$$-S < m_s < S$$

$$|s, m_s\rangle = \left\{ |\underset{\uparrow \atop S}{\tfrac{1}{2}}, \underset{\uparrow \atop m_s}{\tfrac{1}{2}}\rangle, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \right\}.$$

$$S^2 |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar^2 \cdot \tfrac{1}{2}\left(\tfrac{1}{2}+1\right) |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$

$$S^2 \cdot \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \frac{3}{4}\hbar^2 \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$S^2 \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = \frac{3}{4}\hbar^2 \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$S_z \left|\tfrac{1}{2}, \underset{\substack{\uparrow \\ m_s = \frac{1}{2}}}{\tfrac{1}{2}}\right\rangle = \frac{1}{2}\hbar \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

$$S_z \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = -\frac{1}{2}\hbar \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

Reprezentarea matricială :

S^2	$\lvert\frac{1}{2}, \frac{1}{2}\rangle$	$\lvert\frac{1}{2}, -\frac{1}{2}\rangle$
$\langle\frac{1}{2}, \frac{1}{2}\rvert$	$\frac{3}{4}\hbar^2$	0
$\langle\frac{1}{2}, -\frac{1}{2}\rvert$	0	$\frac{3}{4}\hbar^2$

$$\boxed{\begin{gathered} S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ S^2 = \frac{3}{4}\hbar^2 \cdot \mathbb{1}_2 \end{gathered}}$$

S_+	$\lvert\frac{1}{2}, \frac{1}{2}\rangle$	$\lvert\frac{1}{2}, -\frac{1}{2}\rangle$
$\langle\frac{1}{2}, \frac{1}{2}\rvert$	0	$\hbar$
$\langle\frac{1}{2}, -\frac{1}{2}\rvert$	0	0

$$\boxed{S_+ = \hbar \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

$$S_+ \lvert s\, m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s+1)}\ \lvert s, m_s+1\rangle$$

$$S_- \lvert s\, m_s\rangle = \hbar \sqrt{s(s+1) - m_s(m_s-1)}\ \lvert s, m_s-1\rangle$$

S_-	$\lvert\frac{1}{2}, \frac{1}{2}\rangle$	$\lvert\frac{1}{2}, -\frac{1}{2}\rangle$
$\langle\frac{1}{2}, \frac{1}{2}\rvert$	0	0
$\langle\frac{1}{2}, -\frac{1}{2}\rvert$	$\hbar$	0

$$\boxed{S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\boxed{S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

S_z	$\lvert \frac{1}{2}, \frac{1}{2}\rangle$	$\lvert \frac{1}{2}, -\frac{1}{2}\rangle$
$\langle \frac{1}{2}, \frac{1}{2}\rvert$	$\frac{\hbar}{2}$	0
$\langle \frac{1}{2}, -\frac{1}{2}\rvert$	0	$-\frac{\hbar}{2}$

$$\boxed{S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

În reprezentarea matricială asociem vectorii proprii

$$\left| \frac{1}{2}, \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left| \frac{1}{2}, -\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>Completitudinea</u>: $\displaystyle\sum_{m_s=-\frac{1}{2}}^{\frac{1}{2}} |s\, m_s\rangle\langle s, m_s| = \mathbb{1}_2$

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle\left\langle\frac{1}{2}, \frac{1}{2}\right| + \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\left\langle\frac{1}{2}, -\frac{1}{2}\right| = \mathbb{1}_2$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \end{pmatrix} = \mathbb{1}_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2 \quad \checkmark$$

(relația de completitudine este satisfăcută)

<u>Ortonormarea</u>:

$$\left\langle \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}, \frac{1}{2}\right\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \quad \checkmark$$

$$\left\langle \frac{1}{2}, \frac{1}{2} \middle| \frac{1}{2}, -\frac{1}{2}\right\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \quad \checkmark$$

$$\left\langle \frac{1}{2} -\frac{1}{2} \middle| \frac{1}{2}\ \frac{1}{2}\right\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad \checkmark$$

$$\left\langle \frac{1}{2}, -\frac{1}{2} \middle| \frac{1}{2}, -\frac{1}{2}\right\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \quad \checkmark$$

$$\langle s\, m_s | s'\, m_{s'} \rangle = \delta_{ss'}\, \delta_{m_s m_{s'}}$$

Este convenabil să introducem matricile Pauli definite

$$\boxed{\vec{S} = \frac{\hbar}{2} \cdot \vec{\sigma}}$$

$\vec{S}$ - spinul particulei
$\vec{\sigma}$ - matricile Pauli.

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

↳ Matricile Pauli.

Matricile Pauli au o multitudine de proprietăți:

$$\sigma_j^2 = \mathbb{I}_2 \quad , j = x, y, z$$

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 0 \; ; \quad j \neq k \qquad \text{(anticomutator)}$$

$$\{\sigma_j, \sigma_k\} = 2\,\mathbb{I} \cdot \delta_{jk} \qquad \text{- anticomutare}$$

$$[\sigma_j, \sigma_k] = 2i\, \varepsilon_{jkl}\, \sigma_l$$

$$\varepsilon_{jkl} = \begin{cases} 1, & \sigma(jkl) = 1 \\ -1, & \sigma(jkl) = -1 \\ 0 & j = k, \text{ sau } j = l \text{ sau } k = l \end{cases} \qquad \text{(tensorul Levi-Civita)}$$

$$\sigma_j^{+} = \sigma_j \qquad j = x, y, z$$

$$\mathrm{Tr}\,(\sigma_j) = 0 \qquad j = x, y, z$$

$$\det\,(\sigma_j) = -1 , \quad j = x, y, z$$

$$\sigma_x \cdot \sigma_y \cdot \sigma_z = i\, \mathbb{I}_2$$

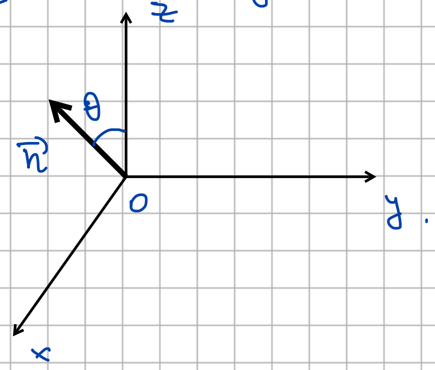
Probleme:

1. Să se găsească vectorii și valorile proprii ale operatorului de spin $\vec{S}$ când acesta este orientat după o direcție $\vec{n}$
$\vec{n} = \sin\theta\, \vec{i} + \cos\theta\, \vec{k}$ în planul Oxz.

$\vec{n}\cdot\vec{S}$.

$$\vec{n} = \sin\theta\, \vec{i} + \cos\theta \cdot \vec{k}$$

$$\vec{S} = \frac{\hbar}{2}\left(\sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k}\right)$$



$$\vec{n}\cdot\vec{S} = \left(\sin\theta\, \vec{i} + \cos\theta\, \vec{k}\right) \times \frac{\hbar}{2}\left(\sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k}\right)$$

$$\vec{n}\cdot\vec{S} = \frac{\hbar}{2}\left(\sigma_x \sin\theta + \sigma_z \cos\theta\right)$$

$$\vec{n}\cdot\vec{S} = \frac{\hbar}{2}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta\right\}$$

$$\boxed{\vec{n}\cdot\vec{S} = \frac{\hbar}{2}\cdot\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}}$$

Am reprezentat matricial operatorul $\vec{n}\cdot\vec{S}$.

La matricea de reprezentare dorim să determinăm valorile proprii și vectorii proprii.

$$\frac{\hbar}{2}\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \cos^2\theta - \sin^2\theta = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$$

valorile proprii $\lambda = \left\{\frac{\hbar}{2}, -\frac{\hbar}{2}\right\}$.

Ecuația pentru vectorii proprii:

$$\frac{\hbar}{2}\cdot\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\cdot\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\sin\theta \cdot a - \cos\theta\, b = b$$

$$a \cdot \sin\theta = b(1+\cos\theta)$$

$$2 \cdot a \cdot \sin\frac{\theta}{2} \cdot \cancel{\cos\frac{\theta}{2}} = 2b \cdot \cos^{\cancel{2}}\frac{\theta}{2}$$

$$a \cdot \sin\frac{\theta}{2} = b \cdot \cos\frac{\theta}{2}.$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$\boxed{|\lambda_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \quad \lambda_+ = \frac{\hbar}{2} \qquad \vec{n}\cdot\vec{S}\,|\lambda_+\rangle = \lambda_+|\lambda_+\rangle}$$

$$\frac{\hbar}{2} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\sin\theta \cdot a = -b(1-\cos\theta)$$

$$2\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2} \cdot a = -2b \cdot \sin^{\cancel{2}}\frac{\theta}{2}$$

$$a \cdot \cos\frac{\theta}{2} = -b \cdot \sin\frac{\theta}{2}.$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$\boxed{|\lambda_-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \quad \lambda_- = -\frac{\hbar}{2} \qquad \vec{n}\cdot\vec{S}\cdot|\lambda_-\rangle = \lambda_-|\lambda_-\rangle}$$

$\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ – vectorul propriu a lui S_z, adică $\vec{n} \parallel Oz$, sau corespunzător $\theta = 0$.

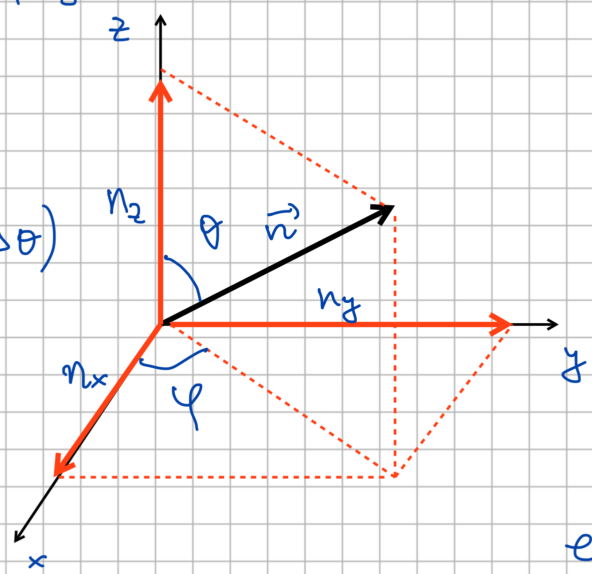
$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\lambda_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} = \cos\frac{\theta}{2}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{cases} |\lambda_+\rangle = \cos\frac{\theta}{2}\cdot\left|\frac{1}{2}\ \frac{1}{2}\right\rangle + \sin\frac{\theta}{2}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \\ |\lambda_-\rangle = -\sin\frac{\theta}{2}\left|\frac{1}{2}, \frac{1}{2}\right\rangle + \cos\frac{\theta}{2}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{cases}$$

Problemă: Hamiltonianul unui sistem este $H = \varepsilon \cdot \vec{\sigma}\cdot\vec{n}$, unde ε este o constantă, cu dimensiune de energie, $\vec{\sigma}$ sunt matricile Pauli iar $\vec{n}$ este un vector cu modul 1 într-o direcție oarecare din spațiu,

$$\begin{cases} \vec{n} = \sin\theta\cdot\cos\varphi\ \vec{i} + \sin\theta\cdot\sin\varphi\ \vec{j} + \cos\theta\cdot\vec{k} \\ \vec{\sigma} = \sigma_x\vec{i} + \sigma_y\vec{j} + \sigma_z\cdot\vec{k} \end{cases}$$



$$H = \varepsilon\left(\sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z\cos\theta\right)$$

$$H = \varepsilon\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\theta\cos\varphi + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\cdot\sin\theta\sin\varphi + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos\theta\right\};$$

$$H = \varepsilon\cdot\begin{pmatrix}\cos\theta & \sin\theta\overbrace{(\cos\varphi - i\sin\varphi)}^{e^{-i\varphi}} \\ \sin\theta\underbrace{(\cos\varphi + i\sin\varphi)}_{e^{i\varphi}} & -\cos\theta\end{pmatrix}$$

$$\boxed{H = \varepsilon\cdot\begin{pmatrix}\cos\theta & \sin\theta\cdot e^{-i\varphi} \\ \sin\theta\cdot e^{i\varphi} & -\cos\theta\end{pmatrix}}$$

↳ reprezentarea Hamiltonianului în baza $\left\{\left|\frac{1}{2}, \frac{1}{2}\right\rangle, \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right\}$

$$\left\langle\frac{1}{2}, \frac{1}{2}\right| H \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \cos\theta \qquad \left\langle\frac{1}{2}\ \frac{1}{2}\right| H \left|\frac{1}{2} - \frac{1}{2}\right\rangle = \sin\theta\, e^{-i\varphi}$$

$$\left\langle\frac{1}{2} - \frac{1}{2}\right| H \left|\frac{1}{2}\ \frac{1}{2}\right\rangle = \sin\theta\, e^{i\varphi} \qquad \left\langle\frac{1}{2} - \frac{1}{2}\right| H \left|\frac{1}{2} - \frac{1}{2}\right\rangle = -\cos\theta$$

Se verifică imediat că $H = H^{+} = (H^{T})^{*}$

1) Să se găsească valorile proprii (energiile sistemului) și vectorii proprii ai lui H.

$$\begin{vmatrix} \varepsilon\cos\theta - \lambda & \varepsilon\sin\theta\, e^{-i\varphi} \\ \varepsilon\sin\theta\, e^{i\varphi} & -\varepsilon\cos\theta - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \varepsilon^2\cos^2\theta - \varepsilon^2\sin^2\theta \underbrace{e^{i\varphi}\cdot e^{-i\varphi}}_{1} = 0$$

$$\lambda^2 = \varepsilon^2(\cos^2\theta + \sin^2\theta) = \varepsilon^2$$

$$\boxed{E_+ = \varepsilon \qquad E_- = -\varepsilon}$$ - valorile proprii ale lui H

Valorile proprii ale Hamiltonianului sunt energiile sistemului.

<u>Vectorul propriu coresp. valorii $E_+ = \varepsilon$</u>

$$\varepsilon\cdot\begin{pmatrix} \cos\theta & \sin\theta\cdot e^{-i\varphi} \\ \sin\theta\cdot e^{i\varphi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \varepsilon\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\varepsilon\cdot\cos\theta\cdot a + \varepsilon\sin\theta\cdot e^{-i\varphi}\, b = \varepsilon\cdot a \qquad |:\varepsilon$$

$$a(1-\cos\theta) = b\sin\theta\, e^{-i\varphi}$$

$$\frac{a}{b} = \frac{\cancel{2}\,\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}\, e^{-i\varphi}}{\cancel{2}\sin^{\cancel{2}}\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}\, e^{-i\varphi}}{\sin\frac{\theta}{2}}$$

$$|E_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$|a|^2 + |b|^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

$$\boxed{\begin{aligned} &E_+ = \varepsilon \text{ - valoare proprie} \\ &H \cdot |E_+\rangle = E_+ |E_+\rangle \\ &|E_+\rangle = \begin{pmatrix} \cos\theta \, e^{-i\varphi} \\ \sin\theta \end{pmatrix} \end{aligned}}$$

Vectorul propriu coresp. valorii $E_- = -\varepsilon$

$$\varepsilon \cdot \begin{pmatrix} \cos\theta & \sin\theta \cdot e^{-i\varphi} \\ \sin\theta \cdot e^{i\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\varepsilon \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\varepsilon \cdot \cos\theta \cdot a + \varepsilon \sin\theta \cdot e^{-i\varphi} b = -\varepsilon \cdot a \qquad |:\varepsilon$$

$$a(1+\cos\theta) = -b \sin\theta \, e^{-i\varphi}$$

$$\frac{a}{b} = -\frac{\sin\frac{\theta}{2} \, e^{-i\varphi}}{\cos\frac{\theta}{2}}$$

$$\boxed{\begin{aligned} &|E_-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \, e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix} \\ &E_- = -\varepsilon \\ &H|E_-\rangle = E_- |E_-\rangle \end{aligned}}$$

Problema. Se consideră un sistem cu momentul cinetic total $j = 1$. În reprezentarea $|j, m\rangle$, componentele operatorului moment cinetic sunt:

$$j_x = \frac{\hbar}{\sqrt{2}} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad j_y = \frac{\hbar}{\sqrt{2}} \cdot \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad j_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

a) Să se determine valorile posibile ptr j_x

b) Să se calculeze $\langle j_z \rangle$, $\langle j_z^2 \rangle$, Δj_z dacă sistemul se găsește în starea cu $j_x = -\hbar$.

a) Valorile posibile ale lui j_x reprezintă setul de valori proprii ale op. $\hat{j}_x$.

$$\det\left(j_x - \lambda \mathbb{1}_3 \right) = 0.$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0.$$

$$-\lambda^3 + \lambda + \lambda = 0. \qquad \lambda(\lambda^2 - 2) = 0.$$

$$\lambda_0 = 0 \qquad \lambda = \pm\cancel{\sqrt{2}} \cdot \frac{\hbar}{\cancel{\sqrt{2}}} = \pm \hbar$$

$$\boxed{\begin{aligned} \lambda_{-1} &= -\hbar \\ \lambda_0 &= 0 \\ \lambda_1 &= \hbar \end{aligned}}$$

– reprezintă setul de valori proprii ale lui j_x, reprezintă valorile posibile ale lui j_x în baza postulatului 2 al M.C.

Vectorul propriu corespunzător v.p $\lambda_{-1} = -\hbar$ $|-1\rangle$

$$\frac{\cancel{\hbar}}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\cancel{\hbar} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\frac{b}{\sqrt{2}} = -a \qquad \Rightarrow \qquad b = -a\sqrt{2} \qquad \frac{a}{b} = -\frac{1}{\sqrt{2}}$$

$$b = -c\sqrt{2}$$

$$\frac{b}{\sqrt{2}} = -c \qquad a = c \qquad \frac{c}{b} = -\frac{1}{\sqrt{2}}$$

$$|-1\rangle = N \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$$

$$|\langle -1|-1\rangle|^2 = N^2 \cdot (1+2+1) = 1 \quad \Rightarrow \quad N^2 = \frac{1}{4}$$

$$N = \frac{1}{2}$$

$$\boxed{|-1\rangle = \frac{1}{2}\cdot\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \qquad j_x|-1\rangle = -\hbar|-1\rangle}$$

starea $|-1\rangle$ este o stare proprie a operatorului j_x, corespunzătoare v.p. $\lambda_{-1} = -\hbar$.

Valoarea medie a lui J_z în starea $|-1\rangle$ este:

$$\langle J_z\rangle = \langle -1|J_z|-1\rangle$$

$$= \frac{\hbar}{4}(-1, \sqrt{2}\ -1)\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} = 0.$$

$$\langle J_z^2\rangle = \langle -1|J_z^2|-1\rangle$$

$$= \frac{\hbar^2}{4}\cdot(-1, \sqrt{2}, -1)\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} = \frac{\hbar^2}{2}.$$

$$\Delta J_z = \sqrt{\langle J_z^2\rangle - \langle J_z\rangle^2} = \frac{\hbar}{\sqrt{2}}.$$

c) Să se repete calculul ~~ptr~~ starea când măsurătoarea lui j_x are ca rezultat valoarea $\lambda_0 = 0$.

$$\boxed{|0\rangle = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad j_x|0\rangle = 0.}$$

Starea $|0\rangle$ este starea proprie a operatorului j_x corespunzătoare v.p. $\lambda_0 = 0$.

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\cdot\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$b = 0$$

$$a + c = 0 \Rightarrow a = -c = 1$$

$$b = 0$$

$$|0\rangle = N\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow |\langle 0|0\rangle|^2 = N^2\cdot 2 = 1 \Rightarrow N = \frac{1}{\sqrt{2}}$$

$$|0\rangle = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\langle J_z\rangle = \langle 0|J_z|0\rangle = \frac{1}{\sqrt{2}}\frac{\hbar}{\sqrt{2}}\cdot(1 \;\; 0 \;\; -1)\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$\langle J_z^2\rangle = \langle 0|J_z^2|0\rangle = \frac{1}{2}\hbar^2\,(1 \;\; 0 \;\; -1)\cdot\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\cancel{2}}\hbar^2\cdot\cancel{2} = \hbar^2$$

$$\Delta J_z = \hbar$$